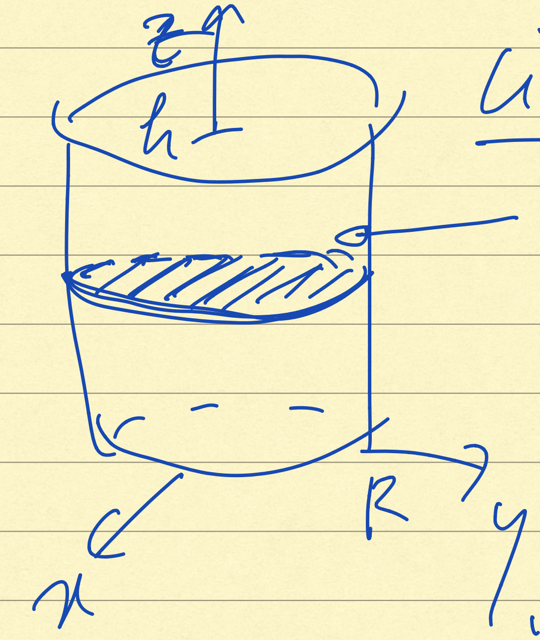
Mudança de Variáveis.

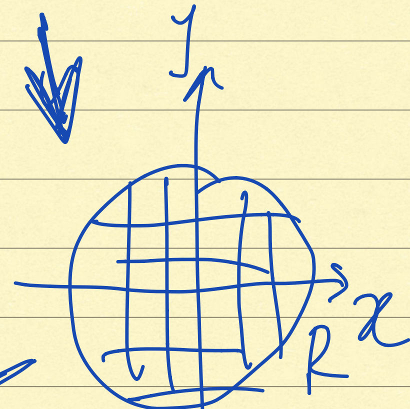
Coordenadas cilíndricas

$$X = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le R^2;\ 0\le z\le h\}$$

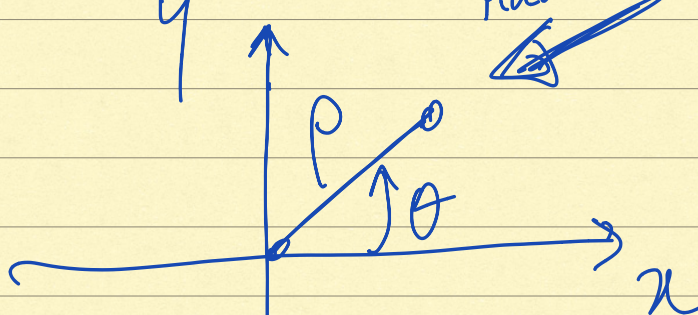
cilindro



z fixo (círculo)



"mudar"



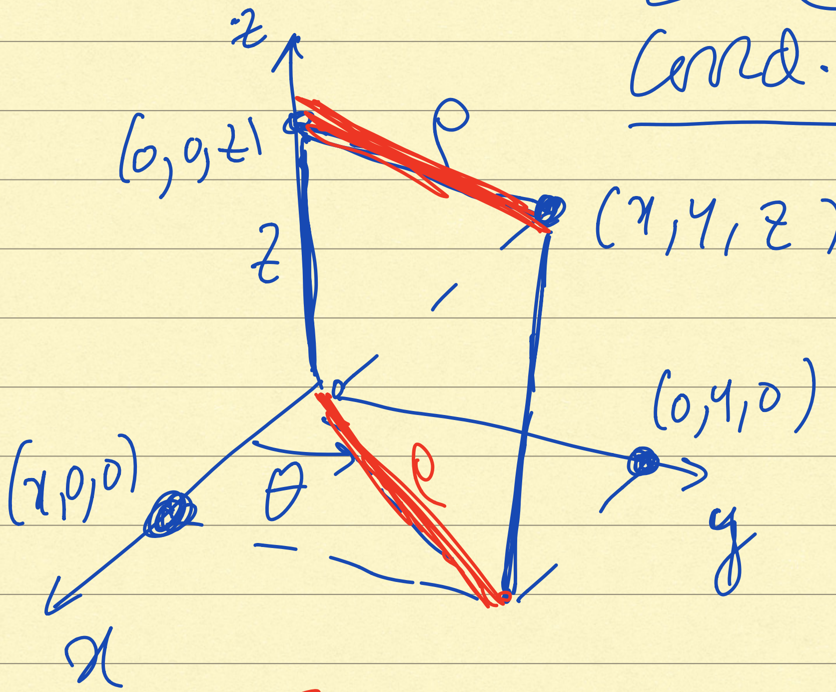
$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \end{cases}$$

$$\boxed{\rho = \sqrt{x^2+y^2}}$$

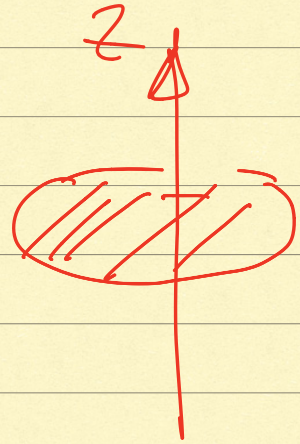
|||

distância ao eixo Oz.

$$(x,y,z) \longleftrightarrow \underbrace{(\rho,\theta,z)}_{\text{coord. cilíndrica}}$$

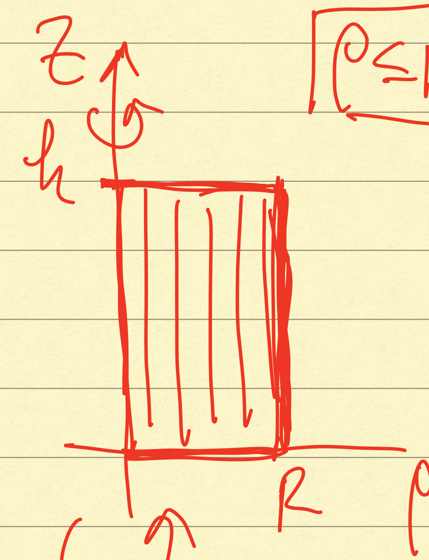


$\boxed{z \text{ fixo}}$ → círculo

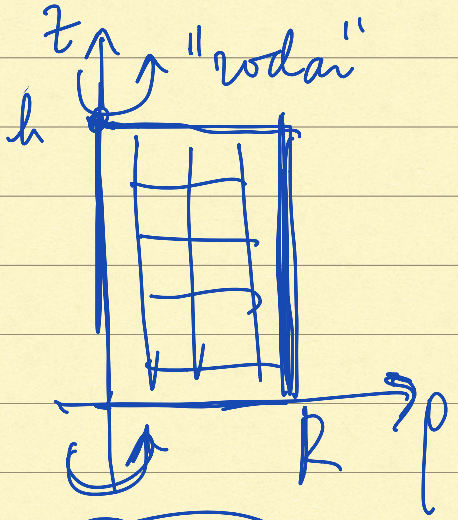


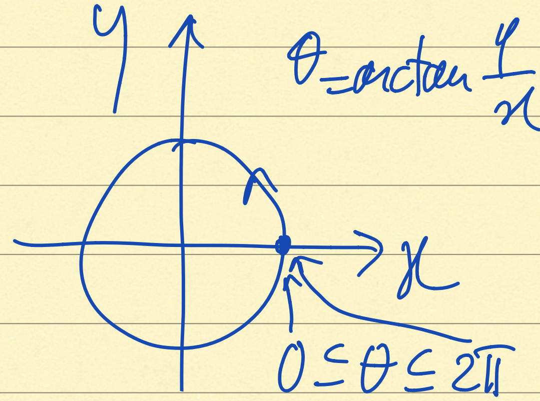
$x^2+y^2 \le R^2$

$\boxed{\rho \le R}$



"rodar"

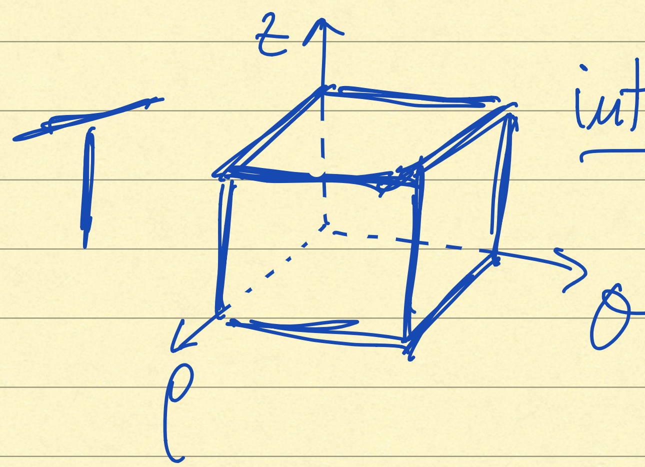




x^2+y^2 $\Rightarrow$ simetria cilíndrica com eixo Oz.

$x^2+y^2 \le R^2$; $0 \le z \le h$

$0 \le \rho \le R$; $0 \le z \le h$; $0 \le \theta \le 2\pi$

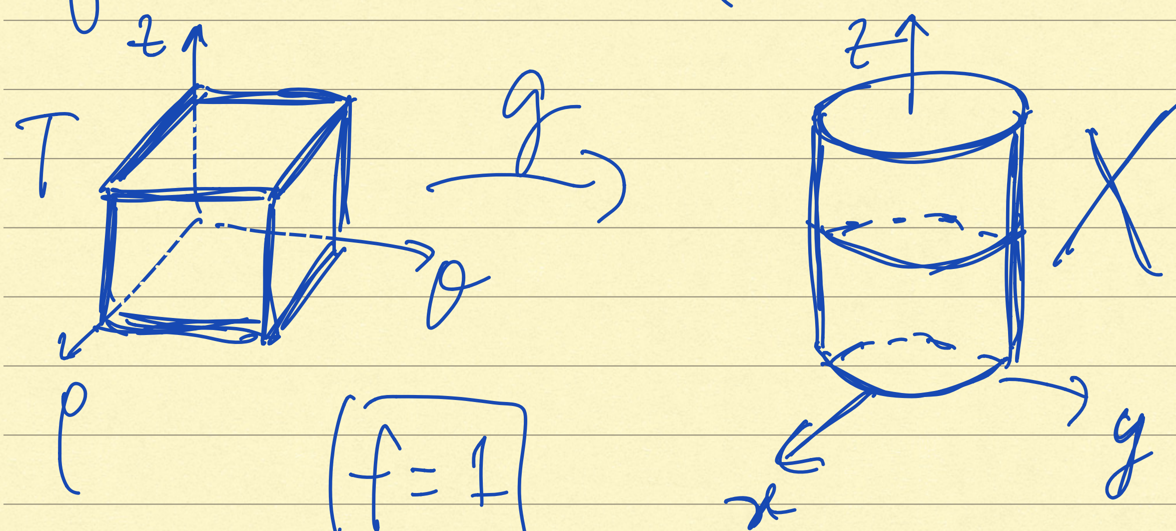


intervalo!!!

$\Downarrow$

integral fácil.

$$g(\rho,\theta,z) = (x,y,z) = (\rho\cos\theta, \rho\,\mathrm{sen}\,\theta, z)$$



$$\boxed{f = 1}$$

$$\iiint_X f = \iiint_T f(g(\rho,\theta,z))\,\rho\, d\rho\, d\theta\, dz$$

$$= \int_0^{2\pi}\left(\int_0^R\left(\int_0^h \rho\, dz\right)d\rho\right)d\theta$$

$$= \dots = \pi R^2 \times h$$

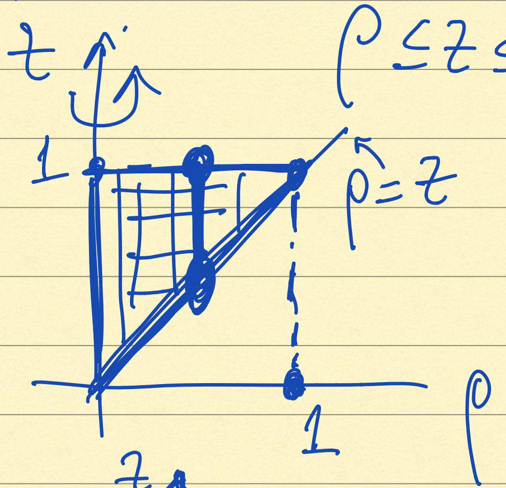
Exemplo: $X = \{(x,y,z) \in \mathbb{R}^3 :$

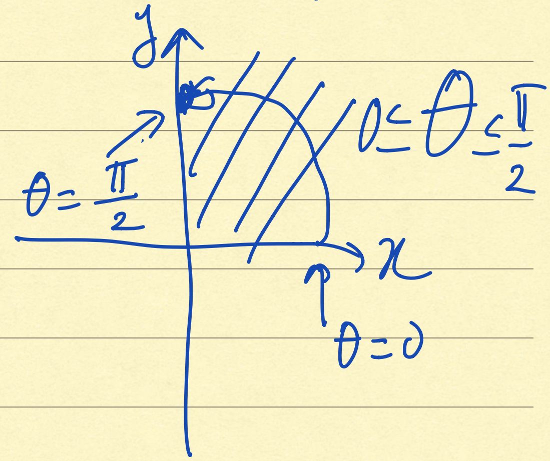
$\sqrt{x^2+y^2} \le z \le 1$

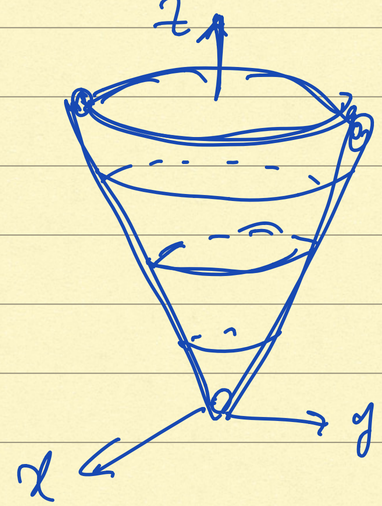
$x, y > 0 \}$

$\boxed{\text{Cone}}$

$\rho \le z \le 1$







$$\mathrm{Vol}_3(X) = \iiint_X 1 =$$

$$= \int_0^{\pi/2} \left(\int_0^1 \left(\int_\rho^1 \rho \, dz \right) d\rho \right) d\theta$$

$=$

$$= \int_0^{\pi/2} \left(\int_0^1 \rho(1-\rho)\, d\rho \right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{3} \right) d\theta = \frac{\pi}{12} \quad //$$

— // —

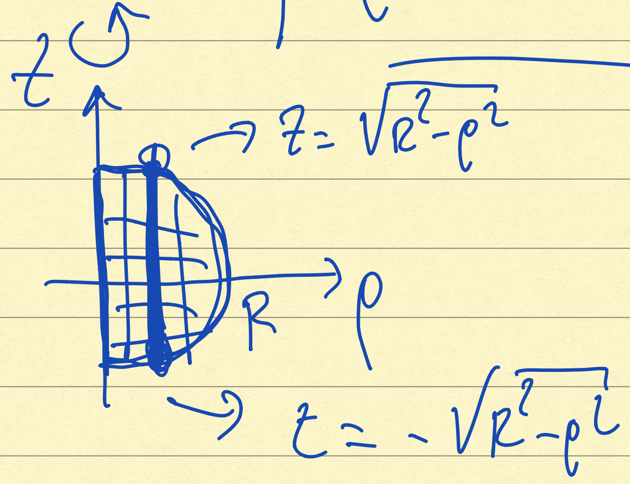
$\rho = \sqrt{x^2+y^2} \geq 0$

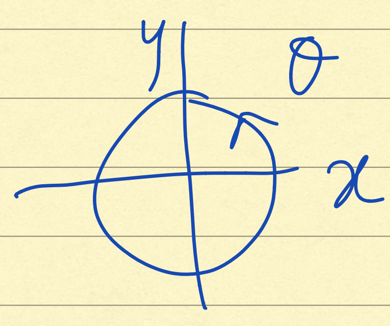
Exemplo:

X : $\underbrace{x^2+y^2}+z^2 \leq R^2$

Bola de raio R

$$\boxed{\rho^2 + z^2 \leq R^2}$$





$$\mathrm{vol}_3(X) = \int_0^{2\pi}\left(\int_0^R\left(\int_{-\sqrt{R^2-\rho^2}}^{\sqrt{R^2-\rho^2}} \rho\, dz\right)d\rho\right)d\theta$$

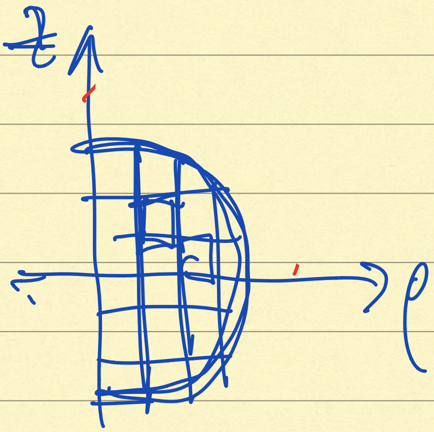
$$= \int_0^{2\pi}\left(\int_0^R \underbrace{2\rho\sqrt{R^2-\rho^2}}_{}\, d\rho\right) d\theta$$

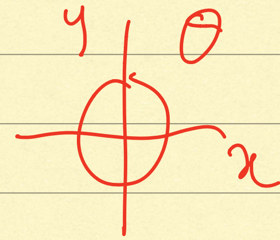
$$\frac{d}{d\rho}\underbrace{\left(R^2-\rho^2\right)}_{}^{3/2} = -\frac{3}{2}\left(R^2-\rho^2\right)^{1/2} 2\rho$$

$$= -3\rho\left(R^2-\rho^2\right)^{1/2}$$

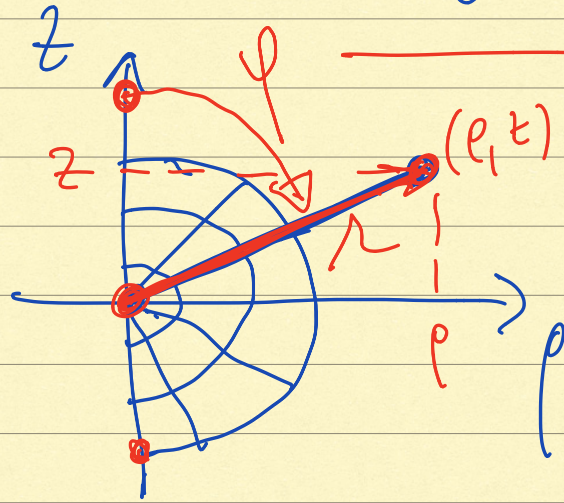
$$\mathrm{vol}_3(X) = +\frac{2}{3}\int_0^{2\pi} R^3\, d\theta = \frac{4}{3}\pi R^3 \quad //$$

$$\rho^2 + z^2 \le R^2$$





<u>mudar</u>



$\varphi \longrightarrow$ "fi"

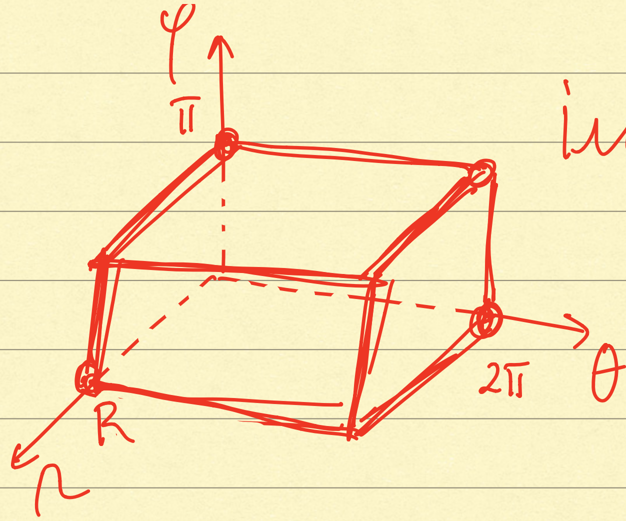
$$r = \sqrt{\rho^2 + z^2}$$

$$(r, \theta, \varphi) \equiv \boxed{\text{Coord. esféricas}}$$

$$\rho^2 = x^2 + y^2$$

$$x^2 + y^2 + z^2 \le R^2 \iff r^2 \le R^2$$

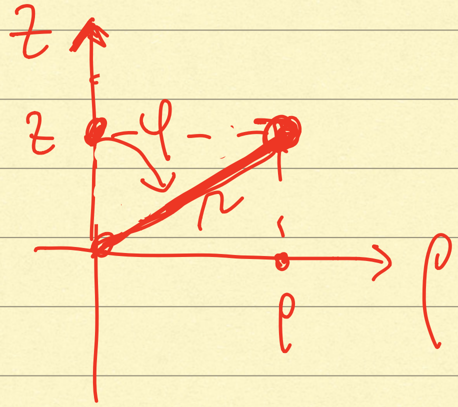
$$\boxed{0 \le r \le R}$$

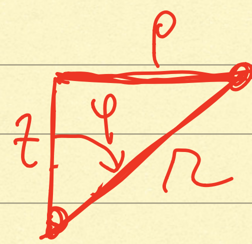


intervalo!

$$g(r,\theta,\varphi) = (x, y, z) = \ ?$$

$$x = \rho \cos\theta$$
$$y = \rho \,\mathrm{sen}\,\theta \longrightarrow \begin{cases} z = r\cos\varphi \\ \rho = r\,\mathrm{sen}\,\varphi \end{cases}$$
$$z = z$$





$$x = \rho \cos\theta = r \,\mathrm{sen}\,\varphi \cos\theta$$

$$y = \rho\, \mathrm{sen}\,\theta = r \,\mathrm{sen}\,\varphi \,\mathrm{sen}\,\theta$$

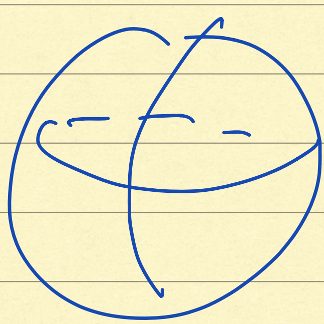
$$z = z = r \cos\varphi$$

$$g(r,\theta,\varphi) = (r\,\mathrm{sen}\,\varphi\cos\theta,\ r\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta,\ r\cos\varphi)$$

$$|\det Dg(r,\theta,\varphi)| = |-r^2\,\mathrm{sen}\,\varphi|$$

$$= \boxed{r^2\,\mathrm{sen}\,\varphi.}$$

$$X: \ x^2+y^2+z^2 \le R^2 \iff \boxed{r^2 \le R^2}$$



$$\mathrm{Vol}_3(X) =$$

$$= \int_0^{2\pi}\left(\int_0^{\pi}\left(\int_0^{R} r^2\,\mathrm{sen}\,\varphi \, dr\right) d\varphi\right) d\theta$$

$$=$$

$$= \cdots = \frac{4}{3}\pi R^3 .$$ //.

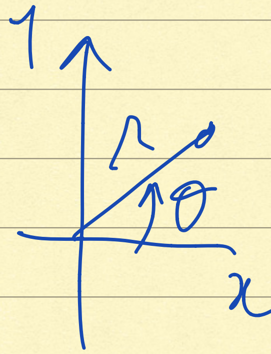
(x, y, z)

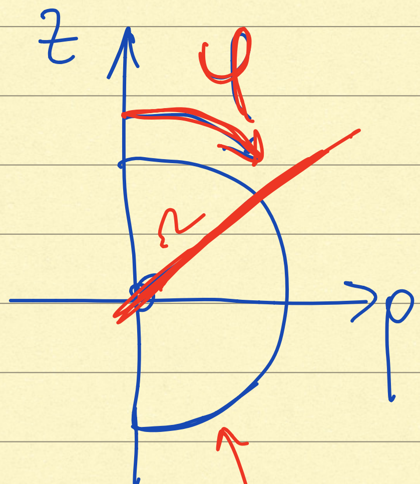
(ρ, θ, z)

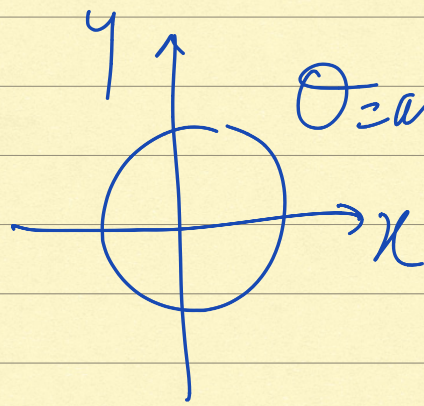
(r, θ, φ)

(x, y)

(r, θ)







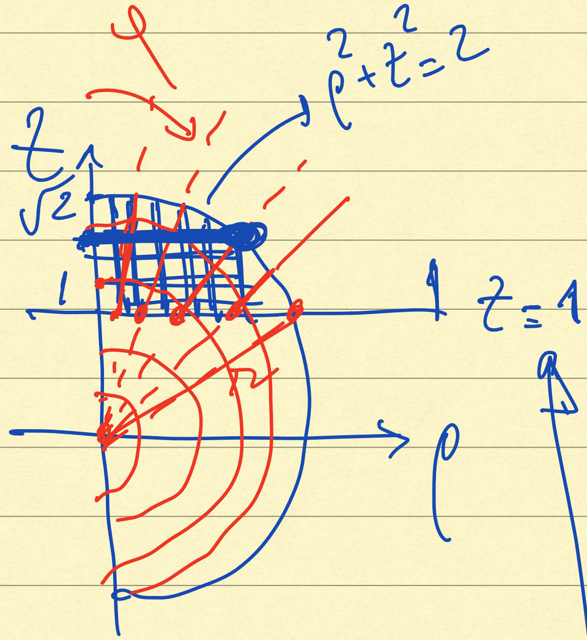
$$\theta = \arctan \frac{y}{x}$$

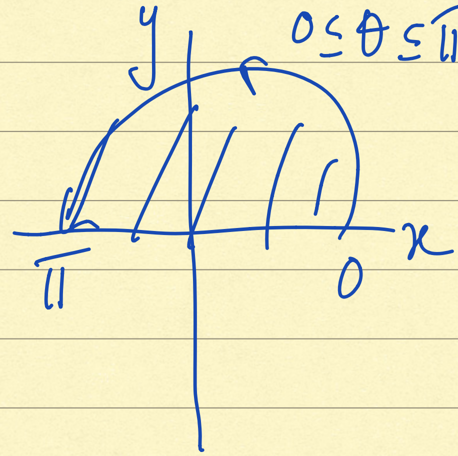
$$\rho^2 + z^2 = r^2$$

$$\begin{cases} \rho = r \operatorname{sen} \varphi \\ z = r \cos \varphi \end{cases}$$

Cilíndricas e esféricas no mesmo diagrama!!

Exemplo: $X: \begin{cases} x^2+y^2+z^2 \leq 2 \\ z \geq 1 \\ y \geq 0 \end{cases}$





"não passa na origem" !!!

(ρ, θ, z)

$$\mathrm{vol}_3(X) = \int_0^{\pi} \left(\int_1^{\sqrt{2}} \left(\int_0^{\sqrt{2-z^2}} \rho \, d\rho \right) dz \right) d\theta =$$

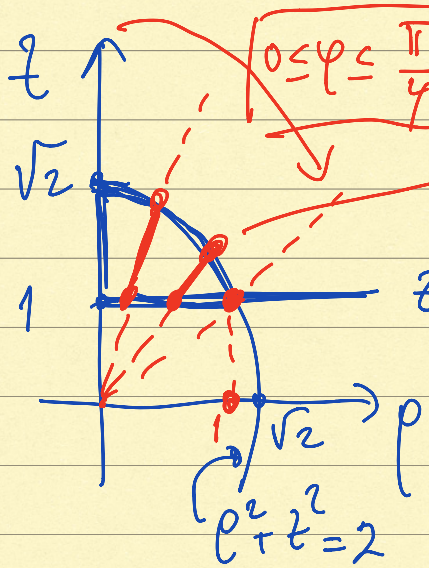
$$= \int_0^{\pi}\left(\int_1^{\sqrt{2}}\frac{1}{2}(2-z^2)\,dz\right)d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\left(2(\sqrt{2}-1)-\frac{1}{3}\left(2^{3/2}-1\right)\right)d\theta$$

$$= \frac{\pi}{2}\left[2(\sqrt{2}-1)-\frac{1}{3}\left(2^{3/2}-1\right)\right]$$ //

<u>Exercício</u>: Calcular $Vol_3(X)$ em coordenadas esféricas (r,θ,φ).

$\boxed{0\le\theta\le\pi}$



$\boxed{0\le\varphi\le\frac{\pi}{4}}$

$r=\sqrt{2}$

$z = 1 = r\cos\varphi \iff r = \dfrac{1}{\cos\varphi}$

$$\boxed{\frac{1}{\cos\varphi}\le r\le\sqrt{2}}$$

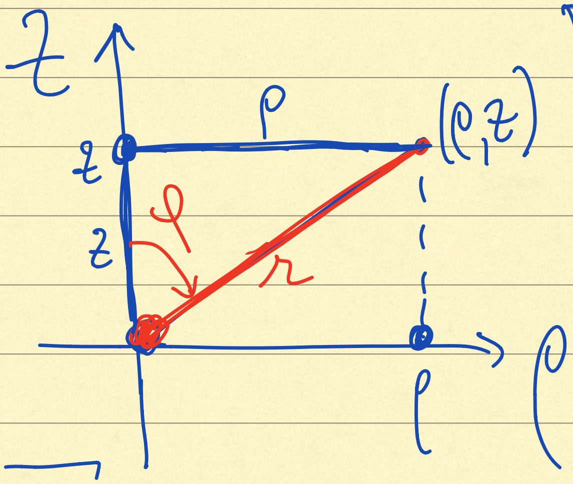
$$Vol_3(X) = \int_0^{\pi}\left(\int_0^{\pi/4}\left(\int_{\frac{1}{\cos\varphi}}^{\sqrt{2}} r^2 \operatorname{sen}\varphi \, dr\right)d\varphi\right)d\theta$$

$= \text{etc} \ldots$

—————— // ——————

Coordenadas cilíndricas e esféricas:

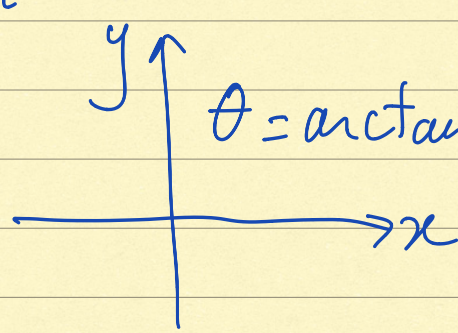
(ρ, θ, z) $\quad\quad$ (r, θ, φ)



$$\boxed{\begin{aligned} r^2 &= \rho^2 + z^2 \\ \rho &= r \operatorname{sen}\varphi \\ z &= r\cos\varphi \end{aligned}}$$

$\rho = \sqrt{x^2+y^2} \equiv$ distância a Oz

$r = \sqrt{x^2+y^2+z^2} \equiv$ distância à origem



$\theta = \arctan \dfrac{y}{x}$